

COUNTING SINGULAR PLANE CURVES VIA HILBERT SCHEMES

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1. INTRODUCTION

Consider the following question. Given a suitable linear series \mathcal{L} on a smooth surface S , how many curves in \mathcal{L} have a given analytic or topological type of singularity? By “suitable” linear series with respect to a type of singularity, we mean that there are finitely many curves with the singularity in the linear series and their codimension is maximal. Our approach to this question is to express the answer as a Chern number of a vector bundle over a compactification of a space linearizing the condition of having the singularity. For example, the condition of having a cusp along a given tangent direction at a given point is linear in the sense that curves in a linear series spanned by two curves with the condition also have the condition. Thus, the projectivized tangent bundle $\mathbb{P}T(S)$ linearizes the condition of having a cusp in S . Note that the closure of the condition of having a cusp in along a given direction is the condition of containing a particular subscheme of S isomorphic to $\text{Spec}(R/(x^2, xy^2, y^3))$ where R is the ring $K[[x, y]]$ and K is the field of definition of S . Generalizing this, the spaces we will use to linearize our conditions will be of the form

$$U(I) = \{a \in \text{Hilb}^d(S) : a \cong \text{Spec}(R/I)\}$$

for ideals I of some finite colength d in R . Section 3 is devoted to this correspondence between ideals and types of singularities and other

conditions on curves. The space $U(I)$ has a natural compactification $C(I)$, its closure in $\text{Hilb}^d(S)$. Letting L be the line bundle corresponding to the divisor of a section of \mathcal{L} , $C(I)$ admits the vector bundle $V_L(I)$ defined in section 4 with the property that sections of \mathcal{L} give sections of the $V_L(I)$ vanishing exactly over those points corresponding to the data of the singularity in question of the curve corresponding to the section. Thus the number of curves having the given types of singularity is the number of places a set of sections of $V_L(I)$ coming from a basis of sections of \mathcal{L} become dependent. This is the Chern number of \mathcal{L} . In order to express this Chern number in terms of the divisor D of a section of L and the Chern classes of the tangent bundle of S , one would like to know the Chern polynomial of $V_L(I)$ pulled back to some space with a known Chow ring, so that the relations in the Chow ring can be used to find the degree of the relevant Chern class. In the case that I can be constructed from the ideals (x, y^3) and (x, y) or from (x, y^2) and (x^2, y) by taking sums, products, and images under the Frobenius morphism if we are working over positive characteristic we can both find the Chow ring of a space dominating $C(I)$ and the Chern classes of $V_L(I)$ in terms of this Chow ring. However, if I is constructed from the ideals (x, y^4) and (x, y) or (x, y^3) and (x^2, y) , we can find the Chow ring of a space dominating $C(I)$, but do not have an algorithm for finding the Chern classes of $V_L(I)$. However, by slightly ad-hoc means one can sometimes or possibly always find these Chern classes. In the last section we give examples of both enumerative problems solved solely by previous results and enumerative problems solved by a mixture of previous results and ad-hoc means.

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2. PRELIMINARIES

The foundation of this paper has been built up in [10] and [11]. We recall some definitions and notation.

We will use the following short hand for denoting monomial ideals. Given a sequence of positive integers s_1, \dots, s_r , we will let $I(n_1, \dots, n_r)$ denote the ideal $(x^r, x^{r-1}y^{n_1}, x^{r-2}y^{n_1+n_2}, \dots)$. Moreover, given a sequence of monomial ideal, I_1, \dots, I_r we let

$$U(I_1, \dots, I_r) = \{(a_1, \dots, a_r) \in U(I_1) \times \dots \times U(I_r) : \exists p \in X \text{ and} \\ \varphi : R \xrightarrow{\sim} \hat{\mathcal{O}}_{X,p} \text{ with } \varphi(I_1, \dots, I_r) = (a_1, \dots, a_r)\}$$

and $C(I_1, \dots, I_r)$ be its closure in the appropriate product of Hilbert schemes. We will say that $C(I_1, \dots, I_r)$ is an *alignment correspondences* with *interior* $U(I_1, \dots, I_r)$. Moreover, we will say that the measuring sequence of I_1, \dots, I_r is A_1, A_2 where A_1 (respectively A_2) is the ideal generated by images of x (respectively y) under automorphisms of R fixing y (respectively x) and sending the I_j 's to themselves. In accordance with [10], although not [11], we let $G(I_1, \dots, I_r)$ be the group of automorphisms of R sending the I_j 's to themselves.

3. CONDITIONS ON CURVES

Definition: Say that a curve C has the *condition* corresponding to an ideal I if C contains a subscheme corresponding to a point of $C(I)$.

We will use the following two lemmas to identify conditions corresponding to ideals.

Definition: Say that a curve C is *generic* with the condition corresponding to an ideal I if I imposes independent conditions on the complete linear series containing C and C is generic among curves in this linear series with this condition.

Lemma 3.1. *Given an ideal I of finite colength in R , the condition on the proper transform of a generic curve with the condition corresponding to I with respect to the blow up of S at the singular point of the curve is the condition corresponding to the quadratic transform of I as defined in [15]. In particular, given a sequence of positive integers n_1, \dots, n_r , the condition corresponding to $I(n_1, \dots, n_r)$ is the closure of the condition of having an r^{th} order point such that if y is a local coordinate for the exceptional divisor of the proper transform of a generic curve C with the condition, then the condition on the proper transform corresponds to the ideal $I(n_1 - 1, \dots, n_r - 1)$.*

Proof: The lemma can be verified by direct computation in coordinate patches. \square

Lemma 3.2. *The condition corresponding to the integral closure of an ideal I is the closure of the condition of having the topological type of singularity of a generic curve with the condition corresponding to I .*

Proof: By [15] the quadratic transform of integral closure of I is the integral closure of the quadratic transform of I . Thus the lemma follows from Lemma 3.1 and induction on the colength of I . \square

Theorem 3.1. *Let I be an ideal with measuring sequence at most $(x, y^3), (x, y)$ (respectively $(x, y^2), (x^2, y)$). Then the boundary of $C(I)$ is equal to the space $C(J)$ where*

$$J = \lim_{t \rightarrow \infty} g(t)(I)$$

and $g(t)$ is the automorphism of R sending x to $x + ty^2$ and fixing y (respectively fixing x and sending y to $y + tx$.)

Proof: The automorphisms of the form $g(t)$ form a set of coset representatives of $G((x, y^2))/G(I, (x, y^2))$. Therefore the boundary of the fiber of $C(I, (x, y^2))$ over $C((x, y^2))$ is the fiber of $C(J, (x, y^2))$ over $C((x, y^2))$ with respect to the projection map. Hence projecting the space $C(I, (x, y^2))$ to $C(I)$, the proposition follows. \square

Definition: Given an ideal I as in Theorem 3.1, we will say that J is the degeneration ideal of I .

Definition: We will call the codimension of curves with the condition given by an ideal I in a linear series on which I imposes independent conditions is *codimension* of the condition, denoted $\text{cod}(I)$.

Lemma 3.3. *The codimension of the condition corresponding to I satisfies*

$$\text{cod}(I) + \dim(C(I)) = \text{col}(I) + \epsilon(I)$$

where $\epsilon(I)$ is the dimension of the locus of subschemes in $C(I)$ contained in a generic curve with the condition given by I .

Proof: Let \mathcal{L} be a linear series for which I imposes independent conditions. Consider the incidence correspondence

$$\{(C, \alpha) \in \mathcal{L} \times C(I) : \alpha \subset C\}.$$

Equating sums of the dimension of the base and fiber with respect to the two projection maps, the lemma follows. \square

Lemma 3.4. *The codimension of the condition of having an analytic type of singularity or a degeneration is the dimension of the versal deformation space of that singularity.*

Proof: The versal deformation space to a singularity can be naturally identified with the normal space to the tangent space of the locus of curves in a linear series such that there is an ideal imposing independent condition on the linear series and generic members have the given type of analytic singularity. \square

Example 3.1.

Let n_1, \dots, n_r be a sequence of increasing positive integers. Then from Lemma 3.1 one can see that the condition of the condition given by the ideal $I(n_1, \dots, n_r)$ is the closure of the topological condition with Enriques diagram a succession of r free vertices of decreasing weights such that there are exactly n_i vertices of weight at most $r + 1 - i$.

Example 3.2.

A generic curve with the condition given by the ideal (x^a, y^b) has the singularity of topological type $x^a + y^b$. The integral closure of the ideal (x^a, y^b) is the ideal $I(a, b)$ generated by monomials $x^c y^d$ with $3c + d \geq b$. Hence this ideal gives the closure of the condition of having the singularity of topological type $x^a + y^b$ and has measuring sequence $(x, y^{\lfloor \frac{b}{a} \rfloor}), (x, y)$.

Example 3.3.

The condition given by the ideal $I(2, 2, 1, 1)$ is strictly in between the closures of the conditions of having topological type and analytic type $x^4 + y^6$. The topological condition is that of having a double cusp. If one blows up at the double cusp of a curve, the proper transform will be a tacnodal curve with both branches tangent to the exceptional divisor. If one then blows up at this point of intersections one gets a quadruple point, with two branches corresponding to the proper transform of the proper transform and two branches corresponding to exceptional divisors. The additional condition corresponding to $I(2, 2, 1, 1)$ is that the four tangent directions of these four branches have cross-ratio -1 . Such curves are of analytic type $x^4 + ax^2y^3 + y^6$ for some constant a . The analytic type varies with a^2 . Unlike the analytic condition above given by a cross ratio, the analytic condition corresponding to a particular choice of a^2 cannot be realized by the configuration of points of intersection of components of the total transform of the curve after some number of blow-ups. It would be interesting to find a geometric way of visualizing such analytic conditions.

Example 3.4.

Theorem 3.1 can be used to glean some information about degenerations of singularities. Given an ideal I as in Theorem 3.1, the condition given by I is that of containing either $\text{Spec}(R/I)$ or $\text{Spec}(R/J)$ where J is the degeneration ideal of I .

For example, degeneration ideal of $I(3, 3, 2)$ is the ideal $I(2, 2, 1, 1)$. This corresponds to the fact that the singularity of topological type $x^3 + y^8$ can degenerate to the singularity of topological type $x^4 + y^6$.

However, only those curves with the additional analytic condition described in the example above can occur.

The sequence giving a condition is not in general unique. For example, the condition corresponding to $1, n$ for any positive integer n is that of being singular. Note that for $n > 2$, a curve with a node will contain two schemes corresponding to points in $C((x^2, xy, y^2))$. The following proposition gives a criterion for when two ideals correspond to the same condition.

Proposition 3.1. *Let I_1 and I_2 be ideals in R with $I_1 \supsetneq I_2$. If*

$$\text{cod}(I_1) = \text{cod}(I_2),$$

then both ideals give the same condition.

Proof: Let \mathcal{L} be the linear corresponding to a sufficiently high tensor power of an ample line-bundle on S . For $i \in \{1, 2\}$, let Γ_i be the incidence correspondence in $\mathcal{L} \times C(I_i)$ as in Lemma 3.3. Then each Γ_i is a vector bundle over $C(I_i)$ and hence irreducible. Therefore the image of the projection map $\pi_1 : \Gamma_i \rightarrow \mathcal{L}$ is irreducible. Since $\pi_1(\Gamma_1)$ contains $\pi_2(\Gamma_2)$ and these two images are both of the same codimension in \mathcal{L} , they must be equal. Suppose by way of contradiction that the $G(I_1)$ orbit of I_2 is not equal to I_1 . Then there is an element $a \in I_1$ that is not in the $G(I_1)$ orbit of I_2 . One can find a curve $C \in \mathcal{L}$ with local equation at a point $p \in S$ equal to the image of a under an isomorphism from R to $\hat{\mathcal{O}}_{S,p}$ up to an element in the image of a high power of the maximal ideal in R . Then C is in $\pi_1(\Gamma_1)$ but not $\pi_2(\Gamma_2)$. It follows that $G(I_1)$ orbit of I_2 is I_1 and hence that I_1 and I_2 correspond to the same conditions. \square

The converse of the previous proposition does not hold in general because one can have ideal corresponding to the same condition, neither of which is contained in the other. However, both will be contained in the ideal of minimal colength corresponding to the condition.

4. VECTOR BUNDLES ON ALIGNMENT CORRESPONDENCES

For each positive integer d let

$$\mathcal{U} \subset \text{Hilb}^d(X) \times X$$

be the universal family over $\text{Hilb}^d(X)$.

Let

$$\pi : \mathcal{U} \rightarrow \text{Hilb}^d(X)$$

and

$$\mu : \mathcal{U} \rightarrow X$$

be the projection maps.

Definition: Given a line bundle L on S , let $V_L(I)$ denote the restriction of $(\pi)_*\mu^*(L)$ to $C(I)$ or by abuse of notation its pullback to any space mapping to $C(I)$.

The fiber in $V_L(I)$ over a point in $C(I)$ is the vector space of germs of sections of L modulo those in the ideal corresponding to the point. If \mathcal{L} is a linear series such that L is the line bundle corresponding to the divisor of a section, then global sections of \mathcal{L} give global sections of $V_L(I)$. These sections of $V_L(I)$ vanish exactly over the points parametrizing subschemes of the divisor of the corresponding section of \mathcal{L} . Hence such a section vanishes if and only if the curve has the condition corresponding to I .

By the following lemma, to find the Chern polynomial of $V_L(I)$, it is enough to find the Chern polynomial of $V(I)$.

Lemma 4.1. *Let A be a space admitting the vector bundle $V(I)$. Then for any line bundle L on S , we have an equality of Chern polynomials*

$$c(V_L(I)) = c(V(I) \otimes (L))$$

over A . Here, abusing notation, we use L to denote its pullback to A .

Proof: Let A' be an extension A over which $V(I)$ has a filtration

$$V(I) = V(I_1) \rightarrow \cdots \rightarrow V(I_r) \rightarrow 0$$

such that the successive kernels are line bundles. Then

$$V_L(I) \rightarrow V_L(I_1) \rightarrow \cdots \rightarrow V_L(I_r) \rightarrow 0$$

is a filtration of $V_L(I)$. From the isomorphisms

$$\text{Ker}(V_L(I_j) \rightarrow V_L(I_{j+1})) \cong \text{Ker}(V(I_j) \rightarrow V(I_{j+1} \otimes L))$$

and the Whitney product formula, we see that the lemma holds over A' . Thus it also holds over A . \square

Proposition 4.1. *Let $C(I_1, \dots, I_r)$ be an alignment correspondence such that there is a monomial ideal I with*

$$I_2 \subset I \subset I_1$$

and

$$\dim(C(I_1, \dots, I_r, I)) - \dim(C(I_1, \dots, I_r)) = \dim(I_1/I_2) - 1.$$

If I has dimension 1 (respectively codimension 1) as a subspace of I_1/I_2 , then the space $C(I_1, \dots, I_r, I)$ is the projectivization of the vector bundle $V(I_1/I_2)$ (respectively $V(I_1/I_2)^$) over $C(I_1, \dots, I_r)$.*

Proof: If I has dimension 1 (respectively codimension 1) as a subspace of I_1/I_2 , then $C(I_1, \dots, I_r, I)$ has a natural embedding in $\mathbb{P}V(I_1/I_2)$ (respectively $\mathbb{P}V(I_1/I_2)^*$). Since both spaces are irreducible and of the same dimension, this embedding is an isomorphism. \square

The Chow rings of the spaces $C((x, y^2), (x, y^3))$ and $C((x, y^2), (x^2, y))$ will be particularly useful for enumerative applications due to the fact that they are universal fiberwise $\text{Aut}(R)$ -equivariant compactifications of the spaces $U((x, y^3))$ and $U((x, y^2), (x^2, y))$ respectively.

Lemma 4.2. [4] *Let I_i denote the ideal (x, y^i) . Let c_1 and c_2 denote the first and second Chern classes of the cotangent bundle of S , respectively. Let h_2 denote the hyperplane class of the projectivization of the cotangent bundle of S . Let h_3 and h'_3 be the hyperplane class of the spaces $C(I_2, I_3)$ and $C(I_2, (x^2, y))$ as the projectivization of the bundles $V(I_2/I_1I_2)$ and $V(I_1/I_1^2)$ respectively. Their Chow rings are given by*

$$A(C(I_2, I_3)) = A(B)[h_3]/(h_3 + 2h_2 + 2c_1)(h_3 - h_2)$$

and

$$A(C(I_2, (x^2, y))) = A(B)[h'_3]/((h'_3)^2 + h'_3c_1 + c_2).$$

Proof: The expression for the second Chow ring follows from the fact that $C((x, y^2)(x^2, y))$ is the projectivization of the pullback of the cotangent bundle on S over B .

To find the Chow ring of $C(I_2, I_3)$ we will use the following two short exact sequences of bundles.

$$(1) \quad 0 \rightarrow V(I_2/I_1^2) \rightarrow V(I_1/I_1^2) \rightarrow V(I_1/I_2) \rightarrow 0$$

$$(2) \quad 0 \rightarrow V(I_1/I_2)^2 \rightarrow V(I_2/I_1I_2) \rightarrow V(I_2/I_1^2) \rightarrow 0$$

Since the vector bundle $V(I_2/I_1^2)$ is the tautological bundle over B ,

$$c_1(V(I_2/I_1^2)) = -h_2.$$

Applying the Whitney product formula to the two above sequences, we get

$$c_1(V(I_2/I_1^2)) = c_1 + h_2$$

and thus

$$c(V(I_2/I_1I_2)) = (1 + 2h_2 + c_2)(1 - h_2),$$

giving us the presentation of $A(C(I_2, I_3))$ as claimed. \square

Proposition 4.2. *Let $V(I/J)$ be a bundle of rank one defined on $C((x, y^3))$ (respectively, $C((x, y^2), (y, x^2))$) where I and J are monomial ideals such that their quotient is generated by $x^a y^b$ as a vector space.*

Let $x^c y^d$ be the monomial generating the degeneration ideal of I over the degeneration ideal of J .

Then we have

$$c_1(V(I/J)) = -ah_2 + b(c_1 + h_2) + (a - c)(h_2 - h_3)$$

(respectively

$$-ah_2 + b(c_1 + h_2) + (a - c)(c_1 + h'_3 + h_2)).$$

Proof: Let I_j be the ideal (x, y^j) . Let I' and J' be the maximum monomial ideals with respect to inclusion with measuring sequence at most I_3, I_1 (respectively $I_2, (x^2, y)$) such that I' is generated over J' by $x^a y^b$ and the degeneration ideal of I' is generated by $x^c y^d$ over the degeneration ideal of J' .

Suppose $a \geq c$. Let L be the line bundle $V(I_3/I_1 I_2)$ (respectively $V(I_1/(x^2, y))$). Then there is a map

$$V(I_2/I_2^2)^c \otimes V(I_1/I_2)^b \otimes L^{a-c} \rightarrow V(I'/J').$$

By Proposition 2.1 of [11] it is an isomorphism.

Similarly, if $c \geq a$, letting L be the line bundle $V(I_2/I_3)$ (respectively $V((x^2, y)/I_1^2)$) the map

$$V(I_2/I_1^2)^a \otimes V(I_1/I_2)^d \otimes L^{c-a} \rightarrow V(I'/J')$$

is an isomorphism.

Thus if the lemma holds for the four line bundles that we called L it always holds. The bundles $V(I_3/I_1 I_2)$ and $V((x^2, y)/I_1^2)$ are the tautological bundles over the spaces C_3 and $C_{2,2}$ respectively and hence have first Chern classes $-h_3$ and $-h'_3$ respectively. Thus applying the Whitney product formula together with the knowledge of the middle elements from Lemma 4.2 to the sequences

$$0 \rightarrow V(I_3/I_1 I_2) \rightarrow V(I_2/I_1 I_2) \rightarrow V(I_2/I_3) \rightarrow 0$$

and

$$0 \rightarrow V((x^2, y)/I_1^2) \rightarrow V(I_1/I_1^2) \rightarrow V(I_1/(x^2, y)) \rightarrow 0$$

we see that these four line bundles also satisfy the lemma. \square

5. SOME CHOW RINGS

In this section we give the Chow rings of some spaces that can be expressed as fiber bundles over the projectivized cotangent bundle of S with toric varieties as fibers. We freely use the material in [5]. Given a ray r , we let $v(r)$ denote the smallest integral point which r passes through.

Lemma 5.1. *Given an exact sequence of vector bundles*

$$0 \rightarrow V_2 \rightarrow V_1 \rightarrow V_3 \rightarrow 0,$$

the projectivization of V_2 inside of the projectivization of V_1 has class

$$c_m(\mathcal{O}_{V_1}(1) \otimes V_3)$$

where m is the rank of V_3 .

Proof: The map from V_1 to V_3 gives a section of $\text{Hom}(\mathcal{O}_{V_1}(-1), V_3)$ which vanishes on the image of V_2 . \square

The above lemma was communicated to me by Mike Roth.

Lemma 5.2. *Let Y be a toric variety of dimension 2 with corresponding fan Δ . Let D_1 be a torus invariant divisor corresponding to a ray r_1 in Δ . Let D_0 and D_2 be the torus invariant divisors intersecting D_1 corresponding to rays r_0 and r_2 just counter-clockwise and just clockwise of r respectively. Then the intersection multiplicity of D_0 and D_1 is*

$$D_0 D_1 = \frac{1}{v(r_1) \wedge v(r_2)}$$

and the self-intersection number of D_1 is given by

$$D_1^2 = \frac{v(r_0) \wedge v(r_2)}{(v(r_0) \wedge v(r_1))(v(r_1) \wedge v(r_2))}.$$

Proof: In the Chow ring of Y , one has the relation

$$\sum_{r_i \in \Delta} (v \cdot v_i) D_i = 0$$

for any vector v . Taking v to be orthogonal to v_0 and intersecting with D_1 we get the relations

$$(v_0 \wedge v_2) D_0 D_1 + (v_1 \wedge v_2) D_1^2 = 0.$$

Similarly, taking v instead to be orthogonal to v_2 , we get the relation

$$(v_2 \wedge v_0) D_2 D_1 + (v_1 \wedge v_0) D_1^2 = 0.$$

It is enough to show that $D_0 D_1$ is as claimed because substituting this into the latter relation, we can solve for the self-intersection number of D_1 . Since D_2 does not effect the intersection of D_0 and D_1 , we can assume that D_1 and D_2 intersect in a smooth point. This implies $D_1 D_2 = v_2 \wedge v_1 = 1$. Using this to simplify the relations above, we see that $D_0 D_1$ is as claimed.

Before stating the main result of this section, we recall some definitions from [11]. Given a sequence of ideals I_1, \dots, I_r with measuring sequence $m(4, 1)$ or $m(3, 2)$, recall that $U(I_1, \dots, I_r)$ is naturally a fiber

over the projectivized tangent bundle of S with fiber isomorphic to $G((x, y^2)/G((x, y^4))$ and $G((x, y^2)/G((x, y^3), (x^2, y)))$ respectively. The normalization of the closure of this fiber is a toric variety. We say that the standard fan of this toric variety is the fan with a ray through $(-1, 0)$ corresponding to the divisor corresponding to automorphisms sending x to $x + ty^2$ for some $t \in K$ and fixing y and a ray through $(0, -1)$ corresponding to automorphisms sending x to $x + ty^3$ and fixing y if the measuring sequence is $m(4, 1)$ and fixing x and sending y to $y + tx$ if the measuring sequence is $m(3, 2)$. Moreover, we say that a ray in a standard fan is a bounding ray if it corresponds to a boundary divisor with only one $G((x, y^2))$ fixed point. It was proved in [11] that in all characteristics but 2, the rays through $(0, 1)$ and $(1, 2)$ are bounding rays if they occur and that there are no other bounding rays. In characteristic 2, $(0, 1)$ is again a bounding ray if it occurs, but any other bounding ray must lie in the interior of the convex cone bounded by the rays through $(1, 2)$ and $(0, -1)$.

Theorem 5.1. *Let Y be an $\text{Aut}(R)$ equivariant compactification of $U((x, y^4))$ (respectively $U((x, y^3), (x, y^2))$) over the space $B = U((x, y^2))$. Let Δ be the standard fan (as defined in [11]) corresponding to the fiber of Y over B . Label the rays in Δ clockwise starting from the ray through $(-1, 0)$ so that i^{th} ray is labelled r_{i-1} . Let (n_i, m_i) be the point of smallest positive distance from the origin in r_i having integral coordinates. Let $r + 2$ be the number of rays in Δ . For $1 \leq i \leq r$, let D_i be the boundary divisor corresponding to r_i . If either the characteristic of K is not 2 or there is no bounding ray in the interior of the cone bounded by the rays through $(1, 2)$ and $(0, -1)$ then the Chow ring of $A(Y)$ is generated over $A(B)$ by the classes of the boundary divisors which we will also denote D_i by abuse of notation and the relations are generated by*

$$D_k^2 = s_k D_k D_{k-1} + D_k(a_{k+1}h_2 + b_{k+1}(c_1 + h_2))$$

for $1 < k \leq r$,

$$D_k^2 = s_k D_k D_{k+1} - D_k(a_{k-1}h_2 + b_{k-1}(c_1 + h_2))$$

for $1 \leq k < r$, and

$$D_i D_j = 0$$

for $|i - j| \geq 2$ where s_k is the self-intersection number of the fiber of D_k and

$$(a_k, b_k) = (m_k - n_k, 2m_k - 3n_k)$$

(respectively

$$(a_k, b_k) = (m_k + n_k, 2m_k + n_k)).$$

Proof: The Chow ring of A is generated over the Chow ring of $U((x, y^2))$ by the boundary divisors because the restrictions of the boundary divisors to the fibers generate the Chow rings of the fibers. If $|i - j| \geq 2$ and $\{i, j\} \neq \{0, r + 1\}$ then the relation $D_i D_j = 0$ follows from the fact that D_i does not intersect D_j . To verify the remaining relations, first we will show that D_i is isomorphic to the projectivization of any bundle $V(I/J)$ such that I and J have measuring sequence at most $(x, y^2), (x, y)$ and I is generated over J by x^{a_i} and y^{b_i} . Then we show that if $i < r$ then the intersection of D_{i+1} with D_i is given by the projectivization of the vector bundle $V(I_1/J)$ where I_1 is generated over J by x^{a_i} . Moreover, we show that if $i > 1$, then the intersection D_{i-1} with D_i is given by the projectivization of the vector bundle $V(I_2/J)$ where I_2 is generated over J by y^{b_i} . Then an application of Lemma 4.2 and Lemma 5.1 give us some of the relations. The remaining relations will be verified through those Y that are projectivizations of staircase bundles.

The function $f : \mathbb{A}^2 - 0 \rightarrow \mathbb{P}^1$ given by

$$f((a, b)) = (a^{m_i}, b^{n_i})$$

extends to a regular function on the fiber of D_i which we will also call f . The map

$$\varphi : D_i \rightarrow \mathbb{P}V(I/J)$$

such that for a point p in the fiber of D_i over (x, y^2) , if $f(p) = (s, t)$ then

$$\varphi(p) = (sx^{a_i} + ty^{b_i}) + J$$

is well defined because it is independent of the choice of x and y . Moreover, it can be shown to be an isomorphism. If $i < r$, the intersection of D_i with D_{i+1} restricted to a fiber is $f^{-1}(1, 0)$ where f is restricted to the fiber of D_i over (x, y^2) . Hence, if $i > 1$

$$\mathbb{P}V(I_1/J) = D_{i+1}D_i$$

and if $i < r$

$$\mathbb{P}V(I_2/J) = D_{i-1}D_i.$$

Let ξ be the hyperplane class of $\mathbb{P}V(I/J)$. By Proposition 4.2

$$A(D_i) = A(B)[\xi]/(\xi - a_i h_2)(\xi + b_i(c_1 + h_2)).$$

Lemma 5.1 gives us the relations

$$(\xi - a_i h_2)D_i = D_i D_{i-1}$$

for $i > 1$ and

$$(\xi + b_i(c_1 + h_2))D_i = D_i D_{i+1}$$

base	rays	J_0	J_1	J_2	J_3
C_3	$(1, n), (0, 1)$	$\underbrace{2, \dots, 2}_n, 3, 0$	$\underbrace{2, \dots, 2}_{n+1}, 1$	$\underbrace{2, \dots, 2}_{n+2}$	$\underbrace{2, \dots, 2}_n, 3, 1$
$C_{2,2}$	$(n, 1), (1, 0)$	$\underbrace{1, \dots, 1}_{n-1}, 0, 3$	$\underbrace{1, \dots, 1}_{n-1}, 0, 2$	$\underbrace{1, \dots, 1}_{n+1}$	$\underbrace{1, \dots, 1}_n, 2$
C_3	$(1, 2), (0, 1)$	4	3	1, 2	1, 3
C_3	$(1, n), (0, 1)$	$\underbrace{2, \dots, 2}_{n-3}, 1, 4$	$\underbrace{2, \dots, 2}_{n-3}, 1, 3$	$\underbrace{2, \dots, 2}_{n-1}$	$\underbrace{2, \dots, 2}_{n-2}, 3$

TABLE 1. Table of Projective Bundles

for $i < r$. Although, there are no global divisors D_0 or D_{r+1} , if r_i is not a bounding ray, there are global divisors $D_i D_{i\pm 1}$. Since the intersection of the fiber of D_i with the coordinate axes is then $G((x, y^2))$ invariant. So, we extend these equations to any i such that r_i is not a bounding ray. Taking the difference of the two equations we obtain

$$D_i(D_{i+1} - D_{i-1}) = D_i(a_i h_2 + b_i(c_1 + h_2)).$$

Hence for $1 < i \leq r$ and r_{i-1} not a bounding ray

$$D_{i-1} D_i^2 = D_{i-1} D_i(a_{i-1} h_2 + b_{i-1}(c_1 + h_2))$$

and for $1 \leq i < r$ and r_{i+1} not a bounding ray

$$D_{i+1} D_i^2 = -D_{i+1} D_i(a_{i+1} h_2 + b_{i+1}(c_1 + h_2)).$$

Since similar relations hold in the Chow ring of the fiber of Y over B , the Chow ring of Y has relations of the form

$$D_i^2 = s_i D_i D_{i-1} + D_i \eta_1$$

for $i > 1$ and

$$D_i^2 = s_i D_i D_{i+1} + D_i \eta_2$$

for $i < r$ where s_i is the self-intersection number of the fiber D_i over B and η is the pullback of a class from B . Multiplying these two equations by D_{i+1} and D_{i-1} respectively, we see that unless r_i is a bounding ray, if $i > 1$,

$$\eta_1 = a_{i+1} h_2 + b_{i+1}(c_1 + h_2)$$

and if $i < r$,

$$\eta_2 = -a_{i-1} h_2 - b_{i-1}(c_1 + h_2).$$

It remains to verify the relations involving D_i^2 for r_i a bounding ray. Since the relations depend on the neighborhood of D_i , it is enough to verify them for some space Y for each possible pair of rays corresponding to adjacent boundary divisors, such that one of the rays is a bounding ray. Any such pair occurs for a space $Y(i, n)$ given by the i^{th}

row of Table 1 as follows. The integer n must be at least $i - 1$, except that $Y((n, 3))$ is independent of n . Let the ideal J_k be as given by the entries in the i^{th} row of the respective column. We define the space $Y(n, i)$ to be the projectivization of the staircase bundle $V(J_1/J_3)$ over the base B' given in the first column. By Theorem 4.1, this is also the space obtained by superimposing B' with the space $C(J_0)$. Thus if $i = 3, 4$ (respectively $i = 1, 2$) then $Y(n, i)$ is a compactification of $U((x, y^4))$ (respectively $U((x, y^3), (x^2, y))$). The second entry in the i^{th} row gives the rays in the fan of the cone of the fiber of $Y(n, i)$ over B corresponding to the two boundary divisors. The first ray corresponds to the pullback of the boundary of B' and the second to the projectivization of the sub-bundle $V(J_2/J_3)$ over B' . By Lemma 5.1, these two divisors have classes $h_4 + c_1(V(J_1/J_2))$ and $h_3 - h_2$ (respectively $h_3 + h_2 + c_1$) where h_4 is the hyperplane class of $Y(n, i)$. Using this to eliminate h_3 and h_4 in the relation from the base C_3 or $C_{2,2}$ and the relation

$$(h_4 + c_1(J_1/J_2))(h_4 + c_1(J_2/J_3)) = 0$$

we recover the relations in the statement of the theorem. \square

Proposition 5.1. *Keeping the notation of the proof of Theorem 5.1, let J_0, J_1, J_2 , and J_3 be the ideals associated to the space $Y(i, n)$ as given in Figure 5 and r_1 and r_2 the two rays given in the i^{th} column of the table. Let D be the divisor of $Y(i, n)$ corresponding to the ray through $(1, 0)$ if $i = 2$ and the ray through $(0, 1)$ otherwise. Then*

$$c_1(V(J_0/J_3)) = -D + c_1(J_1/J_2)$$

and

$$c_1(V(J_1/J_0)) = D + c_1(J_2/J_3).$$

Proof: The bundle $V(J_0/J_3)$ is the tautological bundle over $Y(i, n)$ and thus has first Chern class $-h_4$. Recalling from the proof of Theorem 5.1 that

$$D(r_2) = h_4 + c_1(V(J_1/J_2))$$

and the fact that

$$c_1(J_1/J_0) + c_1(J_0/J_3) = c_1(J_1/J_2) + c_1(J_2/J_3),$$

the proposition follows. \square

6. EXAMPLES

In this section we find the number N_i of curves in a suitable linear series \mathcal{L} on a surface S with the singularity with local equation $x^2 + y^i$ for i from 2 to 8. We will let D denote the divisor of a section of \mathcal{L} and

L the associated line bundle on S . Moreover, we will let I_m denote the ideal (x, y^m) . The ideal

$$B_i = (x^2, xy^{\lfloor \frac{i}{2} \rfloor}, y^i)$$

corresponds to this type of singularity in the sense of section 3. The linear series \mathcal{L} will be of projective dimension $n - 1$, giving n independent sections of the bundle $V_L(B_i)$ over the space $C(B_i)$. Up to scaling, the linear combinations of these sections with zeroes are in bijection with curves in L with the given singularity. Thus the number of these curves is the number of the vector bundle $V_L(B_i)$. This Chern number can be found by finding the Chern class of dimension 0 of $V_L(B_i)$ and then using the Chow ring of the space $C(B_i)$ or some other space dominating to find the degree of this class. Up to $i = 6$, this information is given by Lemma 4.2 and Proposition 4.2. For $i = 7$ and $i = 8$, the relevant Chow rings are given by Theorem 5.1, but we will have to use slightly add-hoc means to find the Chern classes we are interested in.

To find the relevant Chern classes, we will find the Chern classes of the successive quotients in the sequence

$$\begin{aligned} V(I_4^2) &\rightarrow V(I_4^2 + I_1^7) \rightarrow V(I_3^2) \rightarrow V(I_3^2 + I_1^5) \rightarrow V(I_2 I_3) \rightarrow \\ &V(I_2^2) \rightarrow V(I_2^2 + I_1^3) \rightarrow V(I_1 I_2) \rightarrow V(I_1^2) \rightarrow V(I_1) \end{aligned}$$

and then apply the Whitney product formula together with Lemma 4.1.

To find the number N_2 , the number of nodes in a pencil of curves, we need only work over the surface S . Thus N_2 is equal to the second Chern class of $V(I_1^2) \otimes L$. Since this Chern class is already expressed in terms of the Chern classes of the surface and the divisor D , no further substitution is necessary. This example as well as the example of computing N_3 are worked out in detail in [14]. To find N_3 and N_4 , the numbers of cuspidal and tacnodal curves in a suitable linear series \mathcal{L} , we will work over the projectivized cotangent bundle of S . The Chern classes $c_3(V(B_3) \otimes L)$ and $c_3(V(B_4) \otimes L)$ come expressed in terms of divisors pulled back from S and the hyper-plane class h_2 of the cotangent bundle. Using the relation $h_2^2 + c_1 h_2 + c_2 = 0$ to make these classes linear in h_2 , the numbers, N_3 and N_4 are the coefficients of h_2 of these classes. Similarly, to find N_5 and N_6 , we work over the space $Y(I_2, I_3)$. The fourth Chern classes of the bundles $V(B_5) \otimes L$ and $V(B_6) \otimes L$ come expressed in terms of pullbacks of divisors on S , the pullback of the hyperplane class h_2 of the projectivized cotangent bundle of S and the hyperplane class h_3 of $C(I_1, I_2)$. Using the relations in the Chow ring of $C(I_1, I_2)$ to make these Chern classes linear in h_3 and h_2 separately, the numbers N_5 and N_6 are the coefficients of $h_2 h_3$ in these two classes.

Finding the numbers N_7 and N_8 is a bit trickier because we must find the Chern polynomials $c(V(B_7) \otimes L)$ and $c(V(B_8) \otimes L)$ by slightly add-hoc means. In particular, we need to find the Chern classes of the line bundles $V(I_3^2/I_3I_4)$, $V(I_3I_4/B_7)$, and $V(B_7/B_8)$. We will use Table 6 to see where the maps

$$\begin{aligned}\varphi_1 &: V(I_3/I_4)^2 \rightarrow V(I_3^2/I_3I_4), \\ \varphi_2 &: V(I_2/I_3)^3 \rightarrow V(I_3^2/I_3I_4), \\ \varphi_3 &: V(I_4/I_1I_3) \otimes V(I_3/I_4) \rightarrow V(I_3I_4/B_7),\end{aligned}$$

and

$$\varphi_4 : V(I_1I_3/I_1I_4) \otimes V(I_3/I_4) \rightarrow V(B_7/B_8)$$

and then apply Porteous's formula. We will work over the compactification Y of $U(I_4)$ with boundary divisors corresponding to rays through points $(0, 1)$, $(1, 4)$, $(1, 3)$, $(2, 5)$ and $(1, 2)$ in the standard fan of the fiber, since this is the smallest bundle over which all of the vector bundles we will use are defined. Some vector bundles are also defined over smaller spaces. The Chern classes over the smaller spaces can be related the Chern classes over Y via the following lemma.

Lemma 6.1. *Let Z and Z' be two toric varieties of dimension 2 with fans Δ and Δ' such that Δ' is a subdivision of Δ . With respect to the map from Z' to Z compatible with these fans, the pullback of a divisor D corresponding to a ray r_1 in Δ is of the form*

$$\sum_i a_i D_i$$

where i indexes of the rays r_i in Δ' and the D_i 's are the corresponding divisors. The integers a_i can be found as follows. If r_i lies strictly between r_1 and an adjacent ray r_2 in Δ , then if

$$v(r_i) = l_1 v(r_1) + l_2 v(r_2)$$

then a_i is equal to l_1 . If $r_i = r$ then $a_i = 1$. In any other case, $a_i = 0$.

Proof: It is enough to check the lemma in the case that Δ' differs Δ by a single subdivision corresponding to a ray r_3 because the coefficient of the divisor of corresponding to this ray is the same as the coefficient of this ray for any Δ' containing r_3 . The self-intersection number of D is the same as the self-intersection number of its pullback. Writing the pullback of D as $D_1 + qD_3$ we have

$$D^2 = D_1^2 + 2qD_1D_3 + q^2D_3^2.$$

Thus if $D_1^2 = D^2$, then $q = 0$. This verifies the lemma in the cases when $r_1 = r_3$ and when D_1 and D_3 do not intersect. It remains to verify the lemma when r_3 subdivides a cone bounded by r_1 and another ray r_2 .

	(0, 1)	(1, 4)	(1, 3)	(2, 5)	(1, 2)
0, 3	$(x + ay^2) + I_1 I_2$	I_1^2	I_1^2	I_1^2	I_1^2
0, 4	$I_1 I_2$	$I_1 I_2$	$I_1 I_2$	$I_1 I_2$	$(a^2 y^2 - bxy) + I_1^3 + I_2^2$
1, 3	$(xy + ay^3) + I_2^2$	$I_1^3 + I_2^2$	$I_1^3 + I_2^2$	$I_1^3 + I_2^2$	$I_1^3 + I_2^2$
4, 3	$y(x + ay^2)^2 + I_2^3$	$I_2^3 + I_1^5$	$(bx^3 + 2a^3 xy^3) + I_1^2 I_2^2 + I_1^5$	$I_1^3 I_2$	$(a^2 y^4 - 2bxy^3) + I_1^2 I_2^2 + I_1^5$
3, 4	$I_2(xy + ay^3) + I_2^3$	$I_1^2 I_2$	$I_1^2 I_2$	$(3b^2 x^3 + a^5 y^4) + I_1^3 I_2$	I_1^4
1, 4	I_2^2	I_2^2	$(2bx^2 + a^3 y^3) + I_1^2 I_2$	I_1^3	I_1^3
4, 4	I_2^3	$(bx^3 + a^4 y^5) + I_1^2 I_2^2$	$I_1^2 I_2^2 + I_1^5$	$I_1^2 I_2^2 + I_1^5$	$(x^2, (bx - 2ay)^2) + I_1^5$

TABLE 2. Boundary ideals

	c_1	h_2	$D(0, 1)$	$D(1, 4)$	$D(1, 3)$	$D(2, 5)$	$D(1, 2)$
1,1/2,2	2	2					
1,2/2,1	1						
2,1/2,2	3	3					
2,2/2,3	4	4		2	2	4	2
2,3/3,2	2	1					
3,2/3,3	5	5		2	2	4	2
3,3/3,4	6	6	2	6	4	6	3
3,4/4,3	3	2				1	
4,3/4,4	7	7	2	6	5	8	3
0,2/0,3	2	2		1	1	2	1
0,3/1,2		-1		-1	-1	-2	-1
1,2/1,3	3	3		1	1	2	1
0,3/0,4	3	3	1	3	2	3	1
0,4/1,3		-1	-1	-3	-2	-3	-1
1,3/1,4	4	4	1	3	2	4	2

TABLE 3. First Chern classes

Writing $r_3 = q_1 r_1 + q_2 r_2$ and substituting, using Lemma 5.2 the above equation reduces to $(q - q_1)^2 = 0$. Thus $q = q_1$ and we have verified the lemma. \square

Each column of Table 2 corresponds to a boundary divisor of Y and each row corresponds to an ideal as labelled on the top row and left-most column respectively. The sequence m, n corresponds to the ideal x^2, xy^m, y^{m+n} . Let $J(v, I)$ be the entry corresponding to the ray through v and ideal I . This ideal is the projection of the image of I under the generic automorphism to the smallest plane in the Plücker

embedding of the fiber $C(B, I)$ over B containing the boundary divisor corresponding to v .

The map φ_1 drops rank on the boundary divisors D_i such that

$$J(v_i, I_3)^2 \subset J(v, I_3 I_4).$$

Hence by Porteous's formula over Y we have

$$\begin{aligned} c_1(V(I_3^2/I_3 I_4)) &= 2c_1(V(I_3/I_4)) + n_1 D(1, 2) \\ &= 6(c_1 + h_2) + 2D(0, 1) + 6D(1, 4) + 4D(1, 3) + 6D(2, 5) + (2 + n_1)D(1, 2) \end{aligned}$$

where n_1 is a positive integer. Here we have found $c_1(V(I_3/I_4))$ through Proposition 5.1 together with Lemma 6.1. Similarly, one can find the Chern classes of $V(I_4/I_1 I_3)$ and $V(I_1 I_3/I_1 I_4)$ over Y in this way. Following the same steps, Porteous's formula applied to the remaining φ_i 's yields

$$\begin{aligned} c_1(I_3^2/I_3 I_4) &= 6(c_1 + h_2) + n_2 D(0, 1) + (n_3 + 3)D(1, 4) + (n_4 + 3)D(1, 3) + \\ &\quad 6D(2, 5) + 3D(1, 2), \\ c_1(I_3 I_4/B_7) &= 3c_1 + 2h_2 + n_5 D(2, 5), \end{aligned}$$

and

$$\begin{aligned} c_1(B_7/B_8) &= 7(c_1 + h_2) + 2D(0, 1) + 6D(1, 4) + (4 + n_6)D(1, 3) + \\ &\quad (7 + n_7)D(2, 5) + 3D(1, 2). \end{aligned}$$

Comparing the two expressions for $c_1(V(I_3^2/I_3 I_4))$ we see that $n_1 = 1$. Since $C(I_2, I_3^2, B_7)$ does not have a boundary divisor $D(2, 5)$, by Lemma 6.1 the coefficient of $D(2, 5)$ in the first Chern class of $V(I_3^2/B_7)$ as a vector bundle over Y is the sum of the coefficients of $D(1, 3)$ and $D(1, 2)$. By the Whitney product formula $c_1(V(I_3^2/B_7))$ is the sum of $c_1(V(I_3^2/I_3 I_4))$ and $c_1(V(I_3 I_4/B_7))$, we obtain $n_5 = 1$. Similarly, since $C(I_2, I_3^2, I_4^2)$ does not have boundary divisors $D(1, 3)$ and $D(2, 5)$, the coefficient of $D(1, 3)$ in $c_1(V(I_3^2/I_4^2))$ as a class in Y is half the sum of the coefficient of $D(1, 4)$ and the coefficient of $D(1, 2)$. Moreover, the coefficient of $D(2, 5)$ is half the sum of the coefficients of $D(1, 4)$ and three times the coefficient of $D(1, 2)$. By the Whitney product formula, we have

$$c_1(V(I_3^2/I_4^2)) = c_1(V(I_3^2/I_3 I_4)) + c_1(V(I_3 I_4/B_7)) + c_1(V(B_7/I_4^2)).$$

Thus it follows that $n_6 = n_7 = 1$. The first Chern classes we have found or used along the way are listed in Table 3. The leftmost entry $m_1, m_2/m_3, m_4$ of each row signifies that the remaining entries in that row are the coefficients of the divisors listed at the top of each column in the Chern class $c_1(V((x^2, xy_1^m, y^{m_1+m_2}))/((x^2, xy_3^m, y^{m_3+m_4})))$ over Y .

Having found all relevant Chern classes, using the Whitney product formula together with Lemma 4.1, we can find the Chern classes

$c_5(V(B_7) \otimes L)$ and $c_5(V(B_7) \otimes L)$. It remains to calculate the degrees of these Chern classes using the relations for the Chow ring of Y given in Theorem 5.1. Using these relations, these classes can be made free of squares of boundary divisors and linear in h_2 . The respective degrees of the Chern classes are then the sums of the coefficients of terms of the form $h_2 D_i D_{i+1}$. Our results are summarized below.

$$\begin{aligned} N_2 &= 6D^2 + 4Dc_1 + 2c_2 \\ N_3 &= 12D^2 + 12Dc_1 + 2c_1^2 + 2c_2 \\ N_4 &= 50D^2 + 64Dc_1 + 17c_1^2 + 5c_2 \\ N_5 &= 180D^2 + 280Dc_1 + 100c_1^2 \\ N_6 &= 630D^2 + 1140Dc_1 + 498c_1^2 - 60c_2 \\ N_7 &= 2128D^2 + 4368Dc_1 + 2232c_1^2 - 424c_2 \end{aligned}$$

and

$$N_8 = 7272D^2 + 16544Dc_1 + 9548c_1^2 - 2148c_2.$$

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